

## Effect of Relaxation Factor $\omega$ on System of $3 \times 3$ Matrices

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### Abstract

This research work considered the effect of relaxation factor,  $\omega$  on the converging behaviors of successive over relaxation (SOR). To investigate this, three test problems were carefully selected each from  $3 \times 3$  matrices. The experiment was done with an initial guess to exact solutions with relaxation factor,  $\omega$  was chosen at random as; 1.000010, 1.000030, 1.000050, 1.000080 and 1.000090. At the end of the research, 1.000030 or 1.000050 are the best values of relaxation factor,  $\omega$  with least number of iteration, while 1.000010 as the worse value with the highest number of iteration in solving problems involving a class of  $3 \times 3$  matrices.

**Keywords:** System of linear equation, Successive over-relaxation (SOR), relaxation factor

### Introduction

System of linear equation is a collection of two or more linear equations involving the set of variables. The number of variables in the system of equations determines the number of equations in a given system (Poole, 2006). For  $n$ -variables is called the system of  $n$ -linear equation. The divergence behavior of the relaxation factor  $1 < \omega < 2$  is the aim of the paper.

Successive over-relaxation (SOR) method for solving the system of  $n$ -equations is defined by

$$x_i^{(m+1)} = x_i^{(m)} + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1}^n a_{ij} x_j^{(m+1)} - \sum_{j=i}^n a_{ij} x_j^{(m)} \right), \quad i=1,2,\dots,n \quad \text{and} \quad j=1,2,\dots,m, \quad (\text{Hadjidimos, 1999}).$$

SOR iterative method is of industrial standard that is used in solving a matrix system  $Ax=b$  which is applicable in accelerating an interactive solution of the real life problems (mayooran and Elliott, 2016). The SOR rate of convergence strongly depends on the choice of the relaxation factor  $\omega$  (Bailey, 2003). Extensive work has been carried out on finding a good estimate of this factor in the  $(0, 2)$  interval (Kincaid, 2014). Relevant research has shown that for the case when:

- $\omega = 1$ : SOR simplifies to Gauss-Seidel method (Kulrsrud, 1961).
- $\omega \leq 1$  or  $\omega \geq 2$ : SOR fail to converge (Kulrsrud, 1961).
- $\omega > 1$ : SOR used to speed up convergence of a slow-converging process (Young, 1950).
- $\omega < 1$ : help to establish convergence of diverging iterative process (Young, 1950).

For  $0 < \omega < 1$ , the procedure is called under-relaxation method, while  $\omega > 1$ : is called over-relaxation method (Burden and Faires, 2011). The aim of the paper is to investigate the effect of the relaxation factor  $\omega$  on the system of  $3 \times 3$  matrices and also to study the convergence behavior.

## Methodology

For the system of 3x3 linear equation, SOR equations are:

$$x_1^{(m+1)} = x_1^m + \frac{\omega}{a_{11}}(b_1 - a_{11}x_1^m - a_{12}x_2^m - a_{13}x_3^m) \quad (2.1)$$

$$x_2^{(m+1)} = x_2^m + \frac{\omega}{a_{21}}(b_2 - a_{21}x_1^m - a_{22}x_2^m - a_{23}x_3^m) \quad (2.2)$$

$$x_3^{(m+1)} = x_3^m + \frac{\omega}{a_{31}}(b_3 - a_{31}x_1^m - a_{32}x_2^m - a_{33}x_3^m) \quad (2.3)$$

When A is irreducible matrix in the system of a linear equation of the form  $Ax=b$ , with weak diagonal dominance, obviously, both the coefficient of a matrix A and the corresponding diagonal matrix are nonsingular (Shin-liang and Yu-jun, 2016). Successive over-relaxation iteration algorithm is one of the effective method for solving a large-scale sparse matrix equation set, and one first order linear stationary iterative method. The judgment condition of SOR iterative algorithm, depend on the section of the relaxation factor  $\omega$  (Ruicia, 2011).

Identifying the point at which a system of a linear equation converges has become a point of concern in mathematical world as iterations required too much time and energy. This paper tend to study their effect of relaxation factor  $\omega$  and also to identify the best value  $\omega$  in solving the system of 3x3 matrices using SOR.

## Numerical Experiment

### Example 3.1

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 7 \\ 4x_1 - 8x_2 + x_3 &= -21 \\ -2x_1 + x_2 + 5x_3 &= 15 \end{aligned}$$

**3.1.1** When  $\omega = 1.000030$ ,  $m=0$  and  $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = 0 + \frac{1.000030}{4}(7 - 4 \times 0 - 1 \times 0 - 1 \times 0) = 1.750053$$

$$x_2^1 = 0 + \frac{1.000030}{-8}(-21 - 4 \times 1.750053 + 8 \times 0 - 3 \times 0) = 3.500132$$

$$x_3^1 = 0 + \frac{1.000030}{5}(15 + 2 \times 1.750053 - 1 \times 3.500132 - 5 \times 0) = 3.000085$$

$$\therefore x_1^1 = 1.750053, x_2^1 = 3.500132, x_3^1 = 3.500085$$

These values are now the initial and the iterations continue in the same pattern to obtain the following results.

**Table 3.1** iterative results at  $\omega = 1.000030$

M	$x_1^m$	$x_2^m$	$x_3^m$
0	0.000000	0.000000	0.000000
1	1.750053	3.500132	3.000085
2	0.124897	3.812490	2.287439
3	0.225021	3.595294	2.370952

4	0.279318	3.653768	2.380994
5	0.241308	3.638526	2.368818
6	0.248164	3.637389	2.371788
7	0.277706	3.638274	2.371428
8	0.247574	3.638072	2.371415
9	0.247628	3.638097	2.371432
10	0.247618	3.638096	2.371428
11	0.247619	3.638095	2.371429
12	0.247619	3.638095	2.371429

**3.1.2** When  $\omega=1.000050$ ,  $m=0$  and  $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = 0 + \frac{1.000050}{4}(7 - 4 \times 0 - 1 \times 0 - 1 \times 0) = 1.750091$$

$$x_2^1 = 0 + \frac{1.000050}{-8}(-21 - 4 \times 1.750053 + 8 \times 0 - 3 \times 0) = 3.500214$$

$$x_3^1 = 0 + \frac{1.000050}{5}(15 + 2 \times 1.750053 - 1 \times 3.500132 - 5 \times 0) = 3.000115$$

$$\therefore x_1^1 = 1.750091, x_2^1 = 3.500214, x_3^1 = 3.000115$$

These values are now the initial and the iterations continue in the same pattern to obtain the following results.

**Table 3.2** iterative results at  $\omega=1.000050$

M	$x_1^m$	$x_2^m$	$x_3^m$
0	0.000000	0.000000	0.000000
1	1.750091	3.500214	3.000115
2	0.124833	3.812475	2.287403
3	0.224986	3.595259	2.370947
4	0.258450	3.643332	2.374714
5	0.245488	3.638262	2.370543
6	0.247799	3.637853	2.371549
7	0.247649	3.638155	2.371429
8	0.247604	3.638088	2.371424
9	0.247622	3.638095	2.371430
10	0.247619	3.638096	2.371428
11	0.247619	3.638095	2.371429
12	0.247619	3.638095	2.371429

**3.1.3** When  $\omega=1.000080$ ,  $m=0$  and  $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = 0 + \frac{1.000080}{4}(7 - 4 \times 0 - 1 \times 0 - 1 \times 0) = 1.7500140$$

$$x_2^1 = 0 + \frac{1.000080}{-8}(-21 - 4 \times 1.750053 + 8 \times 0 - 3 \times 0) = 3.500350$$

$$x_3^1 = 0 + \frac{1.000080}{5}(15 + 2 \times 1.750053 - 1 \times 3.500132 - 5 \times 0) = 3.000226$$

$$\therefore x_1^1 = 1.750140, x_2^1 = 3.500350, x_3^1 = 3.000226$$

These values are now the initial and the iterations continue in the same pattern to obtain the following results.

**Table 3.3** iterative results at  $\omega = 1.000080$

M	$x_1^m$	$x_2^m$	$x_3^m$
0	0.000000	0.000000	0.000000
1	1.750140	3.500350	3.000226
2	0.124726	3.812473	2.287339
3	0.225055	3.595262	2.370976
4	0.258443	3.643341	2.302704
5	0.263489	3.620257	2.381350
6	0.249597	3.642807	2.371277
7	0.246479	3.637468	2.371098
8	0.247859	3.638099	2.371524
9	0.247594	3.638119	2.371414
10	0.247617	3.638089	2.371429
11	0.247621	3.638096	2.371429
12	0.247619	3.638095	2.371429
13	0.247619	3.638095	2.371429

**3.1.4** When  $\omega = 1.000090$ ,  $m=0$  and  $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = 0 + \frac{1.000090}{4}(7 - 4 \times 0 - 1 \times 0 - 1 \times 0) = 1.751575$$

$$x_2^1 = 0 + \frac{1.000090}{-8}(-21 - 4 \times 1.750053 + 8 \times 0 - 3 \times 0) = 3.503952$$

$$x_3^1 = 0 + \frac{1.000090}{5}(15 + 2 \times 1.750053 - 1 \times 3.500132 - 5 \times 0) = 3.002539$$

$$\therefore x_1^1 = 1.7501575, x_2^1 = 3.503952, x_3^1 = 3.002539$$

These values are now the initial and the iterations continue in the same pattern to obtain the following results.

**Table 3.4** iterative results at  $\omega = 1.000090$

M	$x_1^m$	$x_2^m$	$x_3^m$
0	0.000000	0.000000	0.000000
1	1.751575	3.503952	3.002539
2	0.121912	3.812187	2.285683
3	0.225626	3.594748	2.371378
4	0.258498	3.643560	2.374690
5	0.245426	3.648903	2.368384
6	0.245678	3.635971	2.371079
7	0.248240	3.638277	2.371621
8	0.247525	3.638120	2.371386
9	0.247624	3.638082	2.371433
10	0.247621	3.638098	2.371429
11	0.247618	3.638095	2.371428

12	0.247619	3.638095	2.371429
13	0.247619	3.638095	2.371429

**3.1.5** When  $\omega = 1.000010$ ,  $m=0$  and  $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = 0 + \frac{1.000010}{4}(7 - 4 \times 0 - 1 \times 0 - 1 \times 0) = 1.750002$$

$$x_2^1 = 0 + \frac{1.000010}{-8}(-21 - 4 \times 1.750002 + 8 \times 0 - 3 \times 0) = 3.500005$$

$$x_3^1 = 0 + \frac{1.000010}{5}(15 + 2 \times 1.750002 - 1 \times 3.500005 - 5 \times 0) = 3.000003$$

$$\therefore x_1^1 = 1.750002, x_2^1 = 3.500005, x_3^1 = 3.000003$$

These values are now the initial and the iterations continue in the same pattern to obtain the following results.

**Table 3.5** iterative results at  $\omega = 1.000010$

M	$x_1^m$	$x_2^m$	$x_3^m$
0	0.000000	0.000000	0.000000
1	1.750002	3.500005	3.000003
2	0.124996	3.812499	2.287498
3	0.225001	3.596062	2.370788
4	0.258288	3.643190	2.374677
5	0.245533	3.638270	2.370559
6	0.277793	3.652856	2.380546
7	0.241649	3.638529	2.368954
8	0.248129	3.637422	2.371767
9	0.247703	3.638264	2.371428
10	0.217577	3.623074	2.364216
11	0.253628	3.641720	2.373107
12	0.246293	3.634052	2.371707
13	0.248560	3.638670	2.371690
14	0.247410	3.638089	2.371346
15	0.247641	3.638075	2.371441
16	0.247621	3.6381014	2.371428
17	0.247618	3.638094	2.371428
18	0.247620	3.638096	2.371429
19	0.247619	3.638095	2.371429
20	0.247619	3.638095	2.371429

### Result Discussion

All the iterations started from the initial point of  $x_1^0 = x_2^0 = x_3^0 = 0$  with the value of  $\omega$  as; 0.000030, 0.000050, 0.000080, 0.000090 and 0.000010 respectively. The system diverged at 12 number of iteration on table 3.1 and 3.2, 13 number of iteration on table 3.3 and 3.4 and finally, at 20 number of iteration on table 3.5. At all values of the relaxation factor,  $\omega$  the system diverged to  $x_1^m = 0.247619$ ,  $x_2^m = 3.638095$  and  $x_3^m = 2.371429$ .

## Conclusion

In this paper, we conclude that the rate of convergence of SOR on 3x3 matrices increases as the value of  $\omega$  approaches 2 and decreases as the value of  $\omega$  approaches 1. The numerical analysis shows that the relaxation factor,  $\omega$  of 1.000030 and 1.000050 have the minimum number of 12 iterations, therefore called the best values of  $\omega$ , while 1.000010 is the worse value of the relaxation factor,  $\omega$  with 20 number of iteration.

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